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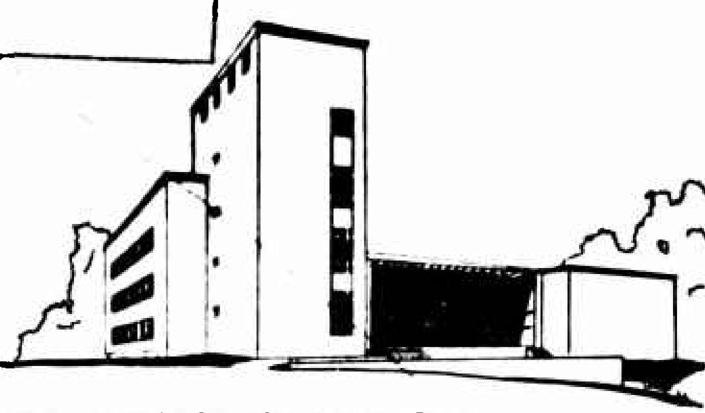

GRADUATE SCHOOL of INDUSTRIAL ADMINISTRATION

William Larimer Mellon, Founder

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OPTIMAL DETECTION OF AN UNKNOWN DISCRETE WAVEFORM WHICH IS RECURRING IN GAUSSIAN NOISE

by

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I. Summary and Introduction

Imbedded in additive Gaussian noise, an unknown acoustic signal $\theta(t)$ is recurring over and over again. The signal consists of a specified number of amplitude modulated pulses of width T sec's. That is, $\theta(t) = \sum_{i=1}^{n} \theta_i \, P[t-(i-1)T]$ where n is known and $P(t) = \begin{cases} 1 & \text{if } 0 \le t < T \\ 0 & \text{otherwise.} \end{cases}$ We do not know these amplitudes θ_i, but suppose $\Sigma\, \theta_i$ and the average rate of recurrence are known. Assume that the recurrence times for the signal are purely random. We wish to determine these recurrence times, i.e., to detect when the signal appears in the noise.

Let $X(t)$ denote the signal plus noise process. We sample $X(t)$ by taking non-overlapping discrete-time records, each being wT sec's in duration. To be more exact, each record consists of w successive observations on $X(t)$ where the observations are taken T sec's apart.

The optimal method for determining whether or not $\theta(t)$ is present in a given record is given by the likelihood-ratio test (Mood [4], Wainstein and Zubakov [5]). If the signal-to-noise ratio is low, then the likelihood-ratio test amounts to observing whether the sum of all the observations in the record exceeds in absolute value a fixed threshold, provided that $\sum_{i=1}^{n} \theta_i = \int \theta(t)dt$ is non-zero. However results are given for the case where $\Sigma\, \theta_i = 0$.

The detection problem is closely connected with the estimation of $\theta(t)$ and its autocorrelation $\psi(\tau) = \int \theta(t+\tau)\, \theta(t)dt$. Hinich [2] discusses this connection and using the tools of large-sample theory, shows that the optimal estimator of ψ is a linear combination of the sample autocorrelation and the square of the sample average of the record again given low signal-to-noise ratio. However the estimation of θ is more involved.

Although this work should be of some use in the problem of active and passive sonar detection of submarines, the initial motivation was from an investigation into an underwater communication system based upon acoustic pulse position modulation, APM. To illustrate this system suppose submarine A is sending a message to submarine B by APM. Sub A repeats an acoustic signal $\theta(t)$ as defined above. The intervals between the repetitions of $\theta(t)$ contain the information which A is sending. The message is coded so that these intervals seem to be purely random in length. However, the average time between recurrences of the signal is at least an order of magnitude longer than the duration of the signal, nT sec's.

Sub B knows n,T, and the pseudo-random code for the intervals between recurrences. If Sub B knows the θ_i, it could detect the times of occurrence by matched filtering (Wainstein and Zubakov [5]). Unfortunately the medium often distorts and delays the pulses, and thus Sub B then doesn't know the shape of the transmitted signal.

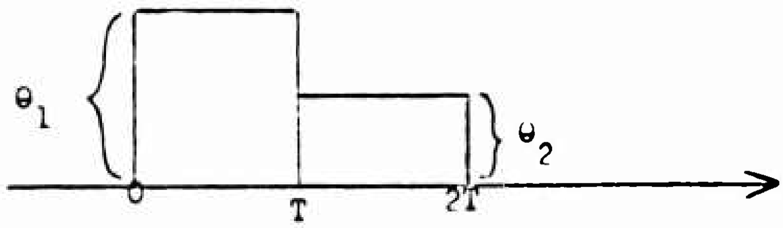
To conclude, let us outline this paper. In Section II we give a formal statement of the problem posed above. In Section III we present (as a Bayes strategy) the likelihood-ratio test for the detection of the recurrence of the waveform. In Section IV we give an approximate likelihood-ratio test for the detection of the waveform when $\Sigma\, \theta_i = 0$. Numerical examples are given for both the case where $\Sigma\, \theta_i \neq 0$ and when it is zero.

II. Statement of Problem

We shall develop a formal statement of the problem. First let us discuss it informally.

We observe a process $X(t)$ which consists of a randomly occurring unknown signal plus noise. The noise process $N(t)$ is assumed to be stationary and Gaussian with mean zero and known covariance. Without loss of generality we may normalize so that the noise has variance $\sigma^2_{N(t)} = EN^2(t) = 1$. Otherwise define $X^*(t) = X(t)/\sigma_{N(t)}$.

The waveform $\theta(t)$ has known length and can be represented as a step function as is illustrated in Figure 1 for $n = 2$.



The parameter T is the pulse width and n is the number of pulses. Thus nT is the duration of the signal. The vector of pulse amplitudes $\theta' = (\theta_1,\ldots,\theta_n)$ is unknown, but assume we know $\theta_1 + \theta_2 + \cdots + \theta_n$.

We assume that the time intervals between repetitions are large compared to the length of the waveform. Let γ be the rate of repetitions of $\theta(t)$. Then $\gamma^{-1}T$ seconds is the average time between waveforms, and thus, γ is small.

We assume that $\|\theta\| = (\sum_{i=1}^{n} \theta_i^2)^{\frac{1}{2}}$ is small compared to the variance of the noise, we can state this in terms of R_θ, the signal-to-noise ratio of $X(t)$. By definition $R_\theta = \gamma \sum_{i=1}^{n} \theta_i^2 / \int_{-\infty}^{\infty} S_N(f)\,df$ where $S_N(f)$ is the spectral density of the noise, $N(t)$. But $\int_{-\infty}^{\infty} S_N(f)\,df = EN^2(t) = \sigma^2_{N(t)} = 1$ by the normalization of N. Thus, $R_\theta = \gamma \sum_{i=1}^{n} \theta_i^2 = \gamma \|\theta\|^2$. Since γ and $\|\theta\|$ are small, R_θ is low.

Now we will discuss the sampling procedure. For fixed integer w, we will take a finite group of w successive discrete observations on $X(t)$; the successive observations being T second apart. That is for each t_i such that $t_1 < t_2 < \cdots < t_m$, we observe

$$X(t_i + T), X(t_i + 2T), \ldots, X(t_i + wT).$$

The intervals between the t_i's $(t_{i+1} - t_i)$ are all substantially greater than wT and nT (the duration of the waveform). Let

$$X^{(i)} = (X(t_i + T), \ldots, X(t_i + wT)) \quad i=1, \ldots, m.$$

We thus have a sample of m vector observations on a w-dimensional random variable.

This sampling scheme may be regarded as opening a sequence of windows of width wT seconds, through which we observe the process at m different stages of time.

There are several different possibilities which can occur when a window is opened. There may be no part of the waveform present during the wT second "look" at $X(t)$. In that case we observe only noise. However, the window may open just as the front part of the waveform is "visible". In that case only the head of the vector θ plus noise is observed. Similarly, we might observe only the tail of θ plus noise, or perhaps the middle of θ plus noise. Incidentally let us suppose that $wT << \gamma^{-1} T$ since we wish to exclude the possibility of catching two successive waveforms in the window.

Since the window has w components and θ has n components, there are $n+w-1$ ways of catching part of θ along with the noise. We can represent these $n+w-1$ possibilities for θ in the window by defining

$$(S_j\theta)' = (\theta_{j+1}, \ldots, \theta_{j+w}) \tag{1}$$

where $\theta_k = 0$ if $k \leq 0$ or $k \geq n+1$. For example $(S_{n-1}\theta)' = (\theta_n 0, \ldots, 0)$ and $(S_{-w+1}\theta)' = (0, 0, \ldots, \theta_1)$. There is no way of knowing in advance which case is occurring. Each of the $n+w-1$ possibilities may be regarded as equally likely. Figure 2 gives an example for $w=3$, $n=2$ where the noise has been removed.

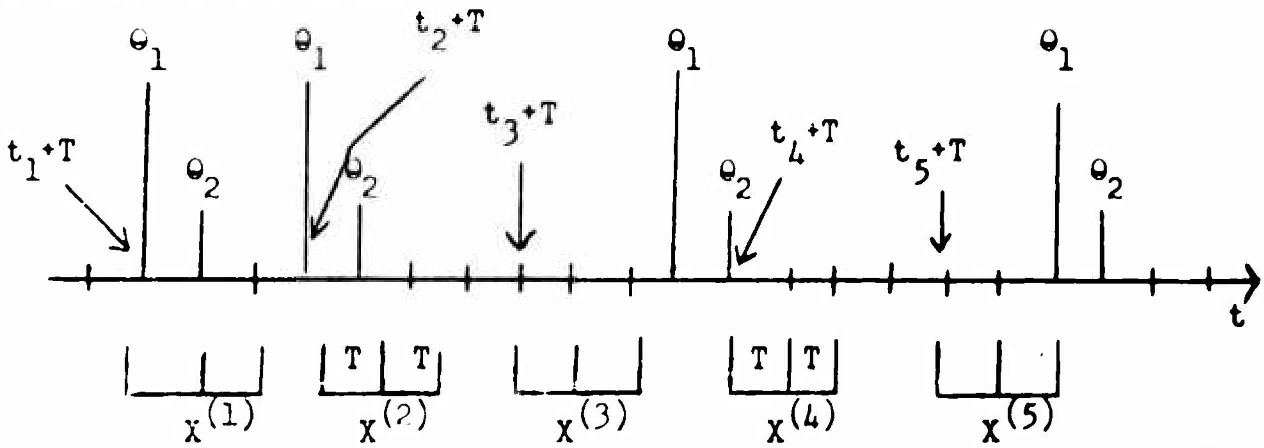


Figure 2. Example of Sampling System with Noise Removed and n=2, w=3.

We will catch some part of θ in the i^{th} window if and only if $\theta(t)$ commences at time t_o: $-nT + t_i + T < t_o < t_i + wT$. Thus, the probability of this event is approximately $(n+w-1)\gamma$ where γ is the recurrence rate. The probability for observing a specific one of the $n+w-1$ possibilities is simply γ .

Since the distances between windows are greater than nT, a single waveform cannot appear in two successive windows. Moreover, suppose that for some τ_o, $EN(t+\tau)N(t) = 0$ for $\tau > \tau_o$. Then if we take the windows further apart than τ_o seconds, the $X^{(i)}$'s are independent. The above restraint on the covariance of the noise holds approximately for many colored noise processes which occur in applications. Of course, for white noise, $\tau_o = 0$.

We then can sum up this discussion with a formal statistical statement of an idealized version of the problem:

We have m independent w-dimensional random vectors $X^{(1)}$, $X^{(2)},\ldots,X^{(m)}$, each identically distributed as X where

$$
(2) \qquad X = \begin{cases} N + S_{-w+1}\theta & \text{with probability } \gamma \\ N + S_{-w+2}\theta & \text{" } \quad \text{" } \quad \text{"} \\ \vdots & \\ N + S_{n-2}\theta & \text{" } \quad \text{" } \quad \text{"} \\ N + S_{n-1}\theta & \text{" } \quad \text{" } \quad \text{"} \\ N & \text{" } \quad \text{" } \quad 1-(n+w-1)\gamma \end{cases}
$$

and $S_j\theta$ is defined in (1).

The vector random variable N has a w-dimensional multivariate normal distribution with mean zero and known, non-singular covariance matrix Σ. We express this by $\mathcal{L}\{N\} = \mathcal{N}(0,\Sigma)$ where $\mathcal{L}\{N\}$ is the distribution function of the random variable N.

Suppose that $\sum_1^n \theta_i$ is either zero or it is a known constant of an order of magnitude greater than $||\theta||^2 = \sum_1^n \theta_i^2$. Assuming that γ and $||\theta||$ are small we desire to test for each record $X^{(i)}$ which of the two following hypotheses are true:

$$
(3) \qquad \begin{aligned} H_o&: X^{(i)} = N^{(i)} \\ H_1&: X^{(i)} = N^{(i)} + S_j\theta \text{ for some } j = -w+1,\ldots, n-1. \end{aligned}
$$

In other words, we wish to determine which records contain part of the waveform along with noise.

III. Statistical Detection of Waveform for $\psi_o = \Sigma\Theta_1 \neq 0$

In this section we will derive the optimal (Bayes) test of the two hypothesis formulated in Section II.

Suppose we have a random variable X from a population with density function $f(x \mid \phi)$. Suppose we wish to test H_o: $\phi = \phi_o$ against H_1: $\phi = \phi_1$ where ϕ_o and ϕ_1 are completely specified. Let h_o and h_1 denote the a-priori probabilities for ϕ_o and ϕ_1, respectively, i.e.

(4)
$$P\{\phi_o \text{ true}\} = h_o$$
$$P\{\phi_1 \text{ true}\} = h_1 = 1-h_o$$

Let L_o denote the loss incurred in rejecting H_o when in fact $\phi = \phi_o$, i.e. the false alarm cost. Let L_1, be the loss in accepting H_o when $\phi = \phi_1$, i.e. the cost in missing the signal. The Bayes strategy is a function of the observed random variable which chooses either H_o or H_1 in such a way as to minimize the expected loss. In Chapter 12, Mood [4] shows that the Bayes strategy is to reject H_o (accept H_1: $\phi = \phi_1$) if

$$\lambda(x) = \frac{f(x|\phi_1)}{f(x|\phi_o)} > k = \frac{h_o L_o}{h_1 L_1} \quad . \tag{5}$$

The random variable X and the parameter ϕ can be vectors. We call $\lambda(x)$ the <u>liklihood-ratio</u>.

We will now deal with the densities of interest in this work. However, to facilitate the algebra we make the one-to-one transformation,

$$Z = \Sigma^{-1} X \tag{6}$$

where Σ^{-1} is the inverse of Σ, the covariance matrix of the Gaussian noise vector N.

From (2) we have

$$(7)\qquad Z = \begin{cases} N^* + \Sigma^{-1} S_j\Theta \text{ with probability } \gamma \text{ for each } j = -w+1,\ldots,n-1 \\ N^* \text{ with probability } \gamma 1-(n+w-1) \end{cases}$$

where $\mathcal{L}\{N^*\} = \mathcal{N}(0, \Sigma^{-1})$.

If the noise $N(t)$ is white, then $EN(t_1)\, N(t_2) = 0$ for $t_1 \neq t_2$. Thus Σ is the identity matrix since we made $\sigma^2_{N(t)} = 1$. Thus $Z = X$.

To restate the hypotheses H_o and H_1 in terms of Z, from (3) we have

$$(8)\qquad \begin{aligned} &H_o: Z = N^* \text{ with a-priori } h_o = 1-(n+w-1)\gamma \\ &H_1: Z = N^* + \Sigma^{-1} S_j\Theta \text{ for some } j \end{aligned}$$

Thus H_o says that only noise is present in the window and H_1 says that some translation of Θ is present.

Let $f(z \mid \Theta)$ be the density of Z given H_1, parameterized by the waveform vector Θ. Therefore from (7) we have

$$(9)\qquad f(z|\Theta) = \frac{1}{n+w-1} \sum_{j=-w+1}^{n-1} n(z|\Sigma^{-1}S_j\Theta, \Sigma^{-1})$$

where

$$n(z|\varphi, C) = (2\pi)^{-\frac{w}{2}} |C|^{-\frac{w}{2}} e^{-\frac{1}{2}(z-\varphi)'C^{-1}(z-\varphi)}$$

is a w-dimensional normal density with mean φ and covariance matrix C. The density of Z given H_o is simply

$$f(z \mid 0) = n(z \mid 0, \Sigma^{-1})$$

Notice that $f(z \mid \Theta)$ is a convex combination of multivariate normal densities, but it is not in general multivariate normal itself. We shall handle it by making Taylor series approximations with Θ in the neighborhood of zero.

By expanding in Taylor series about $\theta = 0$ we have

$$\frac{n(z \mid \Sigma^{-1}\varphi,\ \Sigma^{-1})}{n(z \mid 0,\ \Sigma^{-1})} = 1 + z'\varphi + \frac{1}{2}\varphi'(zz' - \Sigma^{-1})\varphi + \sum_{i,j,k=1}^{w} G_3(z \mid i,j,k)c_3(i,j,k)\varphi_i\varphi_j\varphi_k + \|\varphi\|^4 K^*(z,\varphi) \tag{10}$$

where the c's are constants and

$$\left| K^*(z,\varphi) \right| \le d\, e^{t_1^*|z_1| + \dots + t_w^*|z_w|}$$

for some $t_i^* > 0$ and $d > 0$. Thus $E_o[K^*(Z,\varphi)]^r$ exists and is bounded by some number independent of φ for each $r \ge 0$. With the notation

$$\Sigma^{-1} = (\sigma^{ij})$$

$$G_3(z \mid i,j,k) = z_i z_j z_k - \sigma^{ij}z_k - \sigma^{ik}z_j - \sigma^{jk}z_i \tag{11}$$

$$E_o G_3(z \mid i,j,k) = 0 \quad \text{for all } i,j,k\,.$$

From page 39 of Anderson [1],

$$E_o Z_i Z_j Z_k Z_\ell = \sigma^{ij}\sigma^{k\ell} + \sigma^{ik}\sigma^{j\ell} + \sigma^{i\ell}\sigma^{jk}$$

$$E_o Z_i Z_j = \sigma^{ij}.$$

Moreover all odd moments of the Z_i's are zero. We then have the following orthogonality relationships:

$$E_o Z_i (Z_j Z_k - \sigma^{ik}) = 0 \qquad \text{for all } i,j,k$$

$$E_o Z_i G_3(Z \mid j,k,\ell) = 0 \qquad \text{for all } i,j,k,\ell$$

$$E_o (Z_i Z_j - \sigma^{ij}) G_3(Z \mid k,\ell,m) = 0 \qquad \text{for all } i,j,k,\ell,m.$$

Putting $\varphi = S_j\theta$ in (10) and summing, we have from (5) and (8) that the liklihood ratio $\lambda(z \mid \theta) = \frac{f(z|\theta)}{f(z|0)}$ is,

(12) $$\frac{f(z \mid \theta)}{f(z \mid 0)} = 1 + \frac{1}{n+w-1} \Big[z'(\sum_j S_j\theta) + \frac{1}{2} \sum_j (S_j\theta)'(zz' - \Sigma^{-1})(S_j\theta)$$

$$+ \sum_{i,j,k} G_3(z|i,j,k) O(\|\theta\|^3)$$

$$+ K(z,\theta)\, O(\|\theta\|^4)\Big].$$

$E_0[K(z,\theta)]^r$ exists and is bounded by some number independent of θ for each $r \geq 0$. Moreover the $O(\|\theta\|^3)$ and $O(\|\theta\|^4)$ terms are functions of θ which do not involve z.

Now define the discrete autocorrelation vector $\psi' = (\psi_1, \ldots, \psi_n)$ and the time-average ψ_0 by:

(13)
$$\psi_1 = \frac{1}{2}(\theta_1^2 + \theta_2^2 + \ldots + \theta_n^2) = \frac{1}{2}\|\theta\|^2$$

$$\psi_2 = \theta_1\theta_2 + \theta_2\theta_3 + \ldots + \theta_{n-1}\theta_n$$

$$\psi_3 = \theta_1\theta_3 + \theta_2\theta_4 + \ldots + \theta_{n-2}\theta_n$$

$$\vdots$$

$$\psi_n = \theta_1\theta_n$$

(14) $$\psi_0 = \sum_{i=1}^{n} \theta_i \text{ (DC value of } \theta).$$

Applying (1), (13), and (14) we have

(15) $$\sum_{j=-w+1}^{n-1} S_j\theta = \psi_0 1_w$$

where $1_w' = (\underbrace{1,\ldots,1}_{w})$, and for any symmetric $w \times w$ matrix $A = (a_{ij})$

$$(16) \qquad \frac{1}{2} \sum_{j=-w+1}^{n-1} (S_j\theta)' A(S_j\theta) = \left(\sum_{i=1}^{w} a_{ii}\right)\psi_1 + \left(\sum_{i=1}^{w-1} a_{i,i+1}\right)\psi_2 + \cdots + \left(\sum_{i=1}^{w-n+1} a_{i,i+n-1}\right)\psi_n,$$

where a_{ij} is understood to be zero when i or j is greater than w or less than 1. Thus, if $w < n$, the coefficients of $\psi_{w+1}, \ldots, \psi_n$ vanish and

$$\frac{1}{2} \sum_j (S_j\theta)' A(S_j\theta) = \left(\sum_{i=1}^{w} a_{ii}\right)\psi_1 + \cdots + a_{1w}\psi_w.$$

Applying (15) and (16) to (12) we have the following result.

<u>Lemma 1.</u> Define $Y(z)' = (Y_1(z), \ldots, Y_n(z))$ by

$$(17) \qquad Y_k(z) = \sum_{i=1}^{w-k+1} (z_i z_{i+k-1} - \sigma^{i,i+k-1}), \quad k=1,\ldots,n.$$

Then the liklihood-ratio

$$(18) \qquad \lambda(z\,|\theta) = \frac{f(z\,|\,\theta)}{f(z\,|\,0)} = 1 + \frac{1}{n+w-1}\left[(z'1_w)\psi_o + Y(z)'\psi\right] + \sum_{i,j,k} G_3(z|\,i,j,k)O(\|\theta\|^3) + k(z,\theta)O(\|\theta\|^4)$$

It is understood that $z_i = 0$ if $i \leq 0$ or $i \geq w+1$, and thus if $w < n$

$$Y_{w+1}(z) = \ldots = Y_n(z) = 0.$$

Suppose we know the waveform vector θ. Then from (5) and (8) in Section II, the Bayes test between H_o (only noise) and H_1 (some part of the waveform present) is:

Reject H_o if $\lambda(z|\theta) > k$.

However suppose we know only $\psi_o = \Sigma\theta_i$ but we use an alternative test which says <u>reject</u> H_o if

(19a) $\lambda^*(z|\theta) = 1 + \frac{1}{n+w-1}(z'1_w)\psi_o > k.$ λ^* is the linear approximation of λ. From Hinich [2], we have

$$E_\theta Z'1_w = (1'_w \Sigma^{-1} 1_w) \frac{\psi_o}{n+w-1}$$

$$\mathrm{Var}_\theta Z'1_w = 1'_w \Sigma^{-1} 1_w + \frac{1}{n+w-1} \; O(\|\theta\|^2).$$

Thus we can rewrite (19a) to say <u>reject</u> H_o if

(19b) $$(1'_w \Sigma^{-1} 1_w)^{-1/2} Z' 1_w > \frac{n+w-1}{\psi_o}(k-1).$$

Under H_o, $Y = (1'_w \Sigma^{-1} 1_w)^{-1/2} Z' 1_w$ is a normal random variable with mean zero and variance one. Under H_1, Y is again normal with variance one but with mean

(20) $$(1'_w \Sigma^{i1} 1_w)^{1/2} \frac{\psi_o}{n+w-1}.$$

Let α and α^* be the probabilities for rejecting H_o when in fact it is true (false-alarm) for the λ and λ^* - tests respectively. Let β and β^* be the probabilities of accepting h_o when H_1 is true, i.e. the probabilities of missing the waveform. Then the expected loss E(L) using λ is,

(21a) $$E(L) = \alpha L_o + \beta L_1$$

where L_o is the false-alarm cost and L_1 is the cost of missing θ. Similarly the expected loss $E(L^*)$ using λ^* is,

(21b) $$E(L^*) = \alpha^* L_o + \beta^* L_1.$$

Since the λ - test is Bayes, $E(L) \leq E(L^*)$. But in the Appendix we prove

<u>Theorem 1</u>: The λ^* - test (19b) between H_o and H_1 has expected loss

(22) $$E(L^*) \leq E(L) + (L_o + L_1) \frac{n^2}{(w+n-1)^2}[\psi' D \psi + O(\|\theta\|^6)] + O(\frac{1}{n})$$

where D is the $n \times n$ matrix

(23) $$D = E_o Y(Z)\, Y(Z)'.$$

Example 1: Suppose the noise is white. Thus

$$\Sigma = I_w \text{ and } X = Z$$

Then from (17) we have the autocorrelations,

(24)
$$\begin{aligned} Y_1\,(X) &= (X_1^2 - 1) + \dots + (X_w^2 - 1) \\ Y_2\,(X) &= X_1 X_2 + \dots + X_{w-1}\, X_w \\ &\;\vdots \\ Y_n(X) &= X_1 X_n + \dots + X_{w-n+1}\, X_w \end{aligned}$$

Suppose

$$\theta_1 = \dots = \theta_n = \frac{1}{n}$$

Thus from (13) and (14)

(25) $$\psi' = \frac{1}{n^2}\left(\frac{n}{2}, n-1, n-2, \dots, 1\right)$$

$$\psi'_o = 1$$

Set $w = m$. Thus from (23) and (24)

(26) $$D = \begin{pmatrix} 2n & & & & 0 \\ & n-1 & & & \\ & & n-2 & & \\ 0 & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Thus from (25) and (26)

$$\psi' D \psi \doteq \frac{1}{4}.$$

Now let $L_o = 1$ and $L_1 = 3$. This means that a miss is three times as costly as a false-alarm. This is realistic if there is a-priori knowledge of the pattern of repetitions of θ, allowing us to reduce the false-alarm rate.

Moreover, assume that $1/\alpha = 5n$, i.e. the average distance between occurences of Θ is five times as long as Θ. Then the probability of noise alone in the window is $\frac{4}{5}$. Thus we have

$$k = \frac{1-n\alpha}{n\alpha} \quad \frac{L_o}{L_1} = \frac{4}{3}$$

Now let $n = w = 9$. From (19b) we reject H_o if

$$\frac{1}{3} \sum_1^9 X_i > \frac{n+w-1}{\sqrt{w}} \quad \frac{k-1}{\psi_o} = 1.89. \tag{27}$$

Under H_o, $Y = \frac{1}{3} \sum_1^3 X_i$ has density $n(y \mid 0, 1)$.

$$\alpha^* = P_r\{Y > 1.89 \mid H_o\} = .029 \tag{28a}$$

Under H_1, Y has density $n(y \mid d, 1)$ where

$d = \frac{\sqrt{w}}{n+w-1} \cdot \psi_o = .18$ from (20). Thus

$$\beta^* = P_r\ \{Y < 1.89 \mid H_1\} = .956 \tag{28b}$$

From (21b)

$$E(L^*) = 2.89$$

From (22) and since $\psi' D \psi \doteq \frac{1}{4}$.

$$2.65 < E(L) \leq 2.89 \tag{29}$$

From (28b) we see that the probability of a miss is very high. This is because the detectability—d is small. But this is the best we can do given this sampling scheme. The signal-to-noise ratio is small. If we used matched-filtering with Θ as the filter function, we could not do better.

The test statistic $\frac{1}{\sqrt{w}} \sum_1^w Z_i$ is easy to compute. If we know

ψ_o, we can easily carry out the λ^* - test. For small signal-to-noise ratio it is optimal. It should still be reasonably good even if the signal-to-noise ratio is not too small, especially if we include the second order terms, which involve the waveform autocorrelation terms, ψ_i, and the autocorrelations $Y_i(Z)$.

IV. Detection of Waveform with $\psi_o = 0$.

If $\psi_o = 0$, the λ^* - test is no good. But suppose we know the discrete autocorrelation vector ψ. Furthermore, assume the noise is white. Thus $\Sigma = I$ and $Z = X$. To test between H_o and H_1, let us use the alternative test which says reject H_o if

$$\lambda^{**} (z|\theta) = 1 + \frac{1}{n+w-1} Y(X)'\psi > k, \tag{30a}$$

where the vector $Y(X)$ is given by (24) and $E_o YY' = D$. λ^{**} is the second-order approximation of the likelihood-ratio λ.

From Cochran's theorem and the central limit theorem (Lindgren [3]), under H_o

$$Q = (\psi' D\psi)^{-1/2} \sum_{j=1}^{n} Y_j (X) \psi_j \tag{31}$$

has a normal distribution with mean zero and variance one as $w \to \infty$. Thus we can rewrite (30a) to say reject H_o if

$$Q > (k-1)\ (n+w-1)\ (\psi' D\psi)^{-1/2} \tag{30b}$$

Under H_1 Q is again approximately normal with variance one, for large w, but with mean

$$\frac{1}{n+w-1} (\psi' D \psi)^{1/2}, \tag{32}$$

since from Hinich [2],

$$E_\theta\, Y(X) = \frac{1}{n+w-1} D$$

$$E_\theta\, YY' = D + \frac{1}{n+w-1}\ O(||\theta||^2).$$

We thus can prove in a manner similar to that used for Theorem 1,

Theorem 2: For small θ, the λ^{**} - test (30b) has the property that

$$E(L) \doteq E(L^{**}) = \alpha^{**}L_o + \beta^{**}L_1$$

where α^{**} is the false-alarm probability for the λ^{**} - test and β^{**} is the probability of a miss. In this case we do not specify the error terms. However, by analogy to Theorem 1, the λ^{**} — test should be nearly optimal for reasonable large n.

Example 2: For $w > n$, from (23) and (24)

$$(33) \qquad D = \begin{pmatrix} 2w & & & 0 \\ & w-1 & & \\ & & \ddots & \\ 0 & & & w-n+1 \end{pmatrix}$$

Let $w = 8$ and $n = 4$ with

$$\theta' = \frac{1}{2}(1, -1, -1, 1)$$

From (13) and (14)

$$\psi_o = 0 \text{ and } \psi' = \frac{1}{4}(2, -1, -2, 1).$$

Thus from (33)

$$\psi' D \psi' = \frac{100}{16}.$$

Let $k - 1 = \frac{1}{3}$ as in Example 1. From (30b) and (31) we reject H_o if $Q > 1.47$ where,

$$(34) \qquad Q = \frac{1}{10}\left[2 \sum_1^8 (X_i^2 - 1) - \sum_1^7 X_i X_{i+1} - 2 \sum_1^6 X_i X_{i+2} + \sum_1^5 X_i X_{i+3}\right].$$

Under H_o, Q has density $n(q \mid 0, 1)$.

$$(35a) \qquad \alpha^{**} = P_r\{Q > 1.47 \mid H_o\} = .071$$

Under H_1, Q has density $n(q \mid d, 1)$ where

$$d = \frac{1}{n+w-1}(\psi' D \psi')^{1/2} = .23 \text{ from (32). Thus}$$

$$(35b) \qquad \beta^{**} = P_r\{Q < 1.47 \; H_1\} = .893$$

Thus from Theorem 2,

$$E(L^{**}) = 2.75$$

To conclude, we see from (31) and again in the example with (34) that the test statistic is simply a cross correlation between the sample discrete autocorrelation vector $Y(X)$ and the discrete autocorrelation vector ψ. This cross correlation is fairly easy to compute provided we know the ψ_i. Hinich [2] gives the optimal estimators of the ψ_i, based upon $Y(Z)$ and $Z'1_w$.

In addition we assumed white noise for this section. The case of a general covariance matrix Σ is not a difficult extension, it just complicates the algebra.

Appendix

Proof of Theorem 1: From (18)

$$\alpha = P_r\{\lambda > k|H_o\} \geq P_r\{\lambda^* > k + \frac{1}{n}, |\lambda - \lambda^*| < \frac{1}{n}|H_o\}$$

$$= P_r\{\lambda^* > k + \frac{1}{n}|H_o\} - P_r\{\lambda^* > k + \frac{1}{n}, |\lambda - \lambda^*| > \frac{1}{n}|H_o\}$$

(A1) $$\geq P_r\{(1_w' \Sigma^{-1} 1_w)^{-1/2} \sum_1^w Z_i \geq k'|H_o\} - P_r\{|\lambda - \lambda^*| \geq \frac{1}{n}|H_o\}$$

where,

$$k' = \frac{n+w-1}{(1_w' \Sigma^{-1} 1_w)^{1/2}} \psi_o^{-1}(k-1+\frac{1}{n}).$$

Now $\alpha^* = P_r\{\lambda^* > k|H_o\}$

$$= P\left\{(1_w' \Sigma^{-1} 1_w)^{-1/2} \sum_1^w Z_i \geq k''|H_o\right\}$$

where $k'' = \dfrac{n+w-1}{(1_w' \Sigma^{-1} 1_w)^{1/2}} \psi_o^{-1}(k-1).$

Since $(1_w' \Sigma^{-1} 1_w)^{-1/2} \sum_1^w Z_i$ under H_o is normal with mean zero and variance one, and n is large,

(A2) $$P_r\{(1_w' \Sigma^{-1} 1_w)^{-1/2} \sum_1^w Z_i \geq k'\} = \alpha^* + o(\frac{1}{n}).$$

Moreover by the Tchebysheff inequality, Mood [4],

$$P_r\{|\lambda - \lambda^*| \geq \frac{1}{n}|H_o\} \leq \frac{n^2}{(w+n-1)^2}[\psi' D\psi + O(||\Theta||^6)]$$

since $E_o(\lambda - \lambda^*) = 0$ and from (18),

$$E_o(\lambda - \lambda^*)^2 = \frac{1}{(w+n-1)^2}[\psi' D\psi + O(||\Theta||^6)].$$

Thus from (A1) and (A2),

(A3) $$\alpha \geq \alpha^* - \frac{n^2}{(w+n-1)^2}[\psi' D\psi + O(||\Theta||^6)] + o(\frac{1}{n})$$

Similarly for $\beta = P_r\{\lambda < k|H_1\}$

$$\text{(A4)} \quad \beta \geq \beta^* - \frac{n^2}{(w+n-1)^2} [\gamma' D\gamma + O(\|\theta\|^6)] + o\left(\frac{1}{n}\right)$$

Since from (21b), $E(L^*) = \alpha^* L_o + \beta^* L_1$,

we have the desired result.

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